① $\int\sqrt{1+x^2}\,dx = \int \frac{1+x^2}{\sqrt{1+x^2}}\,dx$

$$\int \frac{1+x^2}{\sqrt{1+x^2}}\,dx = (Ax+B)\cdot\sqrt{1+x^2} + \lambda\int\frac{1}{\sqrt{1+x^2}}$$

$$\frac{1+x^2}{\sqrt{1+x^2}} = A\cdot\sqrt{1+x^2} + (Ax+B)\cdot\frac{1}{2\sqrt{1+x^2}}\cdot 2x + \lambda\,\frac{1}{\sqrt{1+x^2}}$$

$$1+x^2 = A\cdot(1+x^2) + (Ax+B)\cdot x + \lambda$$

$x^2:\ 1 = 2A \quad \Rightarrow A = \frac{1}{2}$

$x^1:\ 0 = B \quad B = 0$

$x^0:\ 1 = A+\lambda \quad \lambda = \frac{1}{2}$

$$\int\sqrt{1+x^2}\,dx = \frac{1}{2}x\cdot\sqrt{1+x^2} + \frac{1}{2}\ln\left|x+\sqrt{1+x^2}\right|$$

② $$\int\frac{\cos x}{\sin^3 x+\sin^2 x}\,dx = \left|\begin{array}{l}\sin x = t\\ \cos x\,dx = dt\end{array}\right| = \int\frac{1}{t^3+t^2}\,dt = \int\frac{1}{t^2(t+1)}\,dt =$$

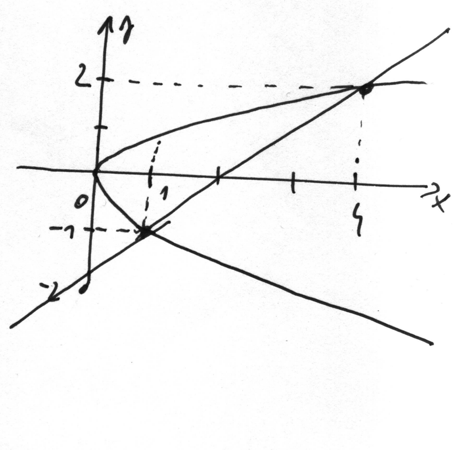
$$= \int\frac{1}{t^2}\,dt - \int\frac{1}{t}\,dt + \int\frac{1}{t+1}\,dt = -\frac{1}{t} - \ln|t| + \ln|t+1| + C =$$

$$= \ln\left|\frac{1+\sin x}{\sin x}\right| - \frac{1}{\sin x} + C$$

③ $\left.\begin{array}{l} y = x-2\\ y^2 = x\end{array}\right\}\quad y+2 = y^2$

$0 = y^2-y-2$

$y_{1,2} = \left\langle\begin{array}{l} 2 \Rightarrow x_1 = 4\\ -1 \Rightarrow x_2 = 1\end{array}\right.$



$S = S_1 + S_2$ $\qquad S_1 = 2 \cdot \int_0^1 \sqrt{x}\,dx = 2 \cdot \left[\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1 = \dfrac{4}{3}$

$$S_2 = \int_1^4 \sqrt{x} - (x-2)\,dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^2}{2} + 2x \right]_1^4 = \frac{2}{3} \cdot \sqrt{4^3} - 8 + 8 -$$

$$- \left(\frac{2}{3} - \frac{1}{2} + 2 \right) = \frac{16}{3} - \frac{2}{3} - \frac{3}{2} = \frac{19}{6}$$

$$S = \frac{4}{3} + \frac{19}{6} = \frac{9}{2}$$

④ $\displaystyle\sum_{n=1}^{\infty} \sin \frac{1}{n^2}$

Porovnávací (limitní): $\displaystyle\lim_{n \to \infty} \frac{\sin \frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{\cos \frac{1}{n^2} \cdot \left(\cancel{-\frac{2}{n^3}} \right)}{\cancel{-\frac{2}{n^3}}} = 1$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ KONVERGUJE $\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty} \sin \frac{1}{n^2}$ KONVERGUJE

⑤ $$\int_1^{1,1} \frac{e^x}{x}\,dx = \int_1^{1,1} \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots}{x}\,dx = \int_1^{1,1} \frac{1}{x} + 1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \dots\,dx$$

$$= \left[\ln x + x + \frac{x^2}{4} + \frac{x^3}{18} + \frac{x^4}{96} + \frac{x^5}{600} + \dots \right]_1^{1,1} = \ln 1,1 + 1,1 + \frac{1,1^2}{4} + \frac{1,1^3}{18} +$$

$$+ \frac{1,1^4}{96} + \frac{1,1^5}{600} + \dots - \left(\ln 1 + 1 + \frac{1}{4} + \frac{1}{18} + \frac{1}{96} + \frac{1}{600} + \dots \right) =$$

$$= 1,589\,689\,849 - 1,317\,638\,889 = 0,272\,050\,960$$

(6.) $xy' - y = x+1$

$y' - \frac{y}{x} = 1 + \frac{1}{x}$

$y' - \frac{y}{x} = 0$

$\frac{1}{y}dy - \frac{1}{x}dx = 0$

$\ln y - \ln x = \ln c$

$y = c \cdot x$

$y = c(x) \cdot x$

$y' = c'(x) \cdot x + c(x)$

$c'(x) \cdot x + c(x) - \frac{c(x) \cdot x}{x} = 1 + \frac{1}{x}$

$c'(x) \cdot x = 1 + \frac{1}{x}$

$c(x) = \int \left(\frac{1}{x} + \frac{1}{x^2}\right) dx = \ln|x| - \frac{1}{x} + c$

$y = \left(\ln|x| - \frac{1}{x} + c\right) \cdot x$

$y = x \cdot \ln|x| - 1 + cx$

$y(1) = 3: \; 3 = 1 \cdot \ln 1 - 1 + c \cdot 1 \Rightarrow c = 4$

$y = x \cdot \ln|x| - 1 + 4x$

(7) $y'' - 2y' = 0$

$\lambda^2 - 2\lambda = 0$

$\lambda(\lambda - 2) = 0$

$\lambda_{1,2} = \begin{cases} 0 \\ 2 \end{cases}$

$y_h = c_1 + c_2 \cdot e^{2x}$

$y'' - 2y' = x$

$y_{p1} = (ax+b) \cdot x$

$y_{p1} = ax^2 + bx$

$y_{p1}' = 2ax + b$

$y_{p1}'' = 2a$

$2a - 2 \cdot (2ax + b) = x$

$2a - 4ax - 2b = x$

$x^1: \; -4a = 1 \Rightarrow a = -\frac{1}{4}$

$x^0: \; 2a - 2b = 0 \Rightarrow b = -\frac{1}{4}$

$y_{p1} = -\frac{1}{4}x^2 - \frac{1}{4}x$

$y'' - 2y' = \sin x$

$y_{p2} = a \cdot \sin x + b \cdot \cos x$

$y_{p2}' = a \cdot \cos x - b \cdot \sin x$

$y_{p2}'' = -a \cdot \sin x - b \cdot \cos x$

$-a \sin x - b \cdot \cos x - 2(a \cos x - b \sin x) =$
$= \sin x$

$\sin x: \; -a + 2b = 1$

$\cos x: \; -b - 2a = 0$

$-b - 2(2b - 1) = 0$

$-5b = -2$

$b = \frac{2}{5}; \; a = -\frac{1}{5}$

$y_{p2} = -\frac{1}{5}\sin x + \frac{2}{5}\cos x$

$y = c_1 + c_2 \cdot e^{2x} - \frac{1}{4}x^2 - \frac{1}{4}x - \frac{1}{5}\sin x + \frac{2}{5}\cos x$